ON QUASI-ELLIPTIC BOUNDARY PROBLEMS

BY
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1. **Introduction.** In this paper we consider the boundary problem

$$P(x, D)u = f \quad \text{in} \quad x_n > 0, \tag{1.1}$$

$$Q_j(x, D)u = g_j \quad \text{on} \quad x_n = 0, 1 \leqq j \leqq r, \tag{1.2}$$

where $P(x, D)$ is a quasi-elliptic operator and $\{Q_j(x, D)\}_{j=1}^r$ is a system of boundary operators satisfying the complementing condition.

First we derive the a priori estimate of the form

$$\|v\|_m \leqq C(\|P(x, D)v\|_0 + \sum_{j=1}^r |Q_j(x, D)v|_{k_j} + \|v\|_0) \tag{1.3}$$

for suitable boundary norms $|Q_j(x, D)v|_{k_j}, j=1, \ldots, r$.

Secondly we shall consider about the hypo-analyticity at the boundary $x_n=0$ for the above problem with the simple boundary operators.

In §2, a definition of a quasi-elliptic operator is given. In §3, a definition of a quasi-elliptic boundary problem is given. §4 and §5 are devoted to derive an a priori estimate (coerciveness estimate) for the case of constant coefficients. In §6 and §7, we shall prove the coerciveness estimate for the case of variable coefficients. In §8 we prove the regularity at the boundary of the solutions of quasi-elliptic boundary problem.

In §§9, 10 and 11 we consider hypo-analyticity at the boundary for quasi-elliptic boundary problems. Theorem 9.2 was suggested by Professor Lions.

The author is deeply indebted to the authors [2], [3] and [12] to make this paper. The author expresses his hearty thanks to Professors Mizohata, Lions, and Kuroda for their valuable suggestions.

2. **Preliminaries.** Let R^n be the n-dimensional Euclidean space whose point is denoted by $(x_1, \ldots, x_n)$. For convenience, set $x=(x_1, \ldots, x_{n-1})$, $y=x_n$ and denote by (x, y) a point of R^n. The half spaces $y>0$ and $y\geqq 0$ are denoted by R^n_+ and $(R^n_+)^a$ respectively.

Let $\alpha=(\alpha_1, \ldots, \alpha_n)$ be a multi-index of nonnegative integers with length $|\alpha|=\alpha_1+\cdots+\alpha_n$. We put $D_j=i^{-1}(\partial/\partial x_j)$, $1\leqq j\leqq n$, $(i^2=-1)$, and

$$D_x = (D_1, \ldots, D_{n-1}), \qquad D_y = D_n, \qquad D = (D_1, \ldots, D_n).$$

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Then the linear differential operators of order m with constant coefficients can be written as

$$(2.1)\qquad P(D) = P(D_x, D_y) = \sum_{|\alpha|\leqq m} a_\alpha D_1^{\alpha_1}\cdots D_n^{\alpha_n},$$

where the coefficients a_α are complex constants. The polynomial corresponding to $P(D_x, D_y)$ is

$$P(\xi, \eta) = \sum_{|\alpha|\leqq m} a_\alpha \xi_1^{\alpha_1}\cdots \xi_{n-1}^{\alpha_{n-1}}\eta^{\alpha_n},$$

where $\xi=(\xi_1,\dots,\xi_{n-1})$.

DEFINITION 2.1. A quasi-elliptic operator of weight q is defined as an operator (2.1) satisfying the following conditions:

$$(2.2)\qquad \begin{gathered} m_j > 0, \quad 1 \leqq i \leqq n, \quad \text{are given integers}, \quad m = \max_{1\leqq j\leqq n} m_j, \\ q_j = m/m_j \geqq 1, \quad \text{and} \quad q = (q_1,\dots,q_n). \end{gathered}$$

$$(2.3)\qquad P(D) = \sum_{\langle\alpha,q\rangle\leqq m} a_\alpha D^\alpha, \quad \text{where} \quad \langle\alpha,q\rangle = \alpha_1 q_1 + \cdots + \alpha_n q_n;$$

the principal part of P with respect to the weight q is $P^0(D)=\sum_{\langle\alpha,q\rangle=m} a_\alpha D^\alpha$.

$$(2.4)\qquad \sum_{1\leqq j\leqq n-1} |\xi_j|^{m_j}+|\eta|^{m_n} \leqq C|P^0(\xi,\eta)| \quad \text{for any} \quad (\xi,\eta)\in R^n.$$

We see that the quasi-elliptic operators are hypo-elliptic. We note that when $m_j=m$ for every j, the quasi-elliptic operators are just the elliptic operators of order m. If $m_n=1$ and $m_j=2$ for $j<n$, we find that the heat equation is quasi-elliptic. Also the p-parabolic equations in the sense of Petrowsky are quasi-elliptic (cf. Friberg [3]).

Let $\tau_1(\xi),\dots,\tau_{m_n}(\xi)$ denote the roots of $P^0(\xi, z)=0$ for each real vector $\xi=(\xi_1,\dots,\xi_{n-1})$. In the case $n>2$ we see from the condition (2.4) that the number r of the roots with the positive imaginary part is independent of $\xi\neq 0$, and in this case we shall say that $P(\xi,\eta)(P(D))$ is of determined type r. In the case of $n=2$ we suppose this root-condition on $P^0(\xi,\eta)$.

3. **Coerciveness inequality** (I). The constant coefficient case.

We consider a quasi-elliptic operator $P(D)$ of weight $q=(m/m_1,\dots,m/m_n)$ and of determined type r $(1\leqq r\leqq m_n)$:

$$(3.1)\qquad P(D) = P(D_x, D_y) = \sum_{\langle\alpha,q\rangle\leqq m} a_\alpha D_1^{\alpha_1}\cdots D_{n-1}^{\alpha_{n-1}} D_n^{\alpha_n}.$$

Here we may assume the coefficient of $D_n^{m_n}$ is equal to 1. The corresponding polynomial of $P(D)$ is

$$P(\xi,\eta) = \sum_{\langle\alpha,q\rangle\leqq m} a_\alpha \xi_1^{\alpha_1}\cdots\xi_{n-1}^{\alpha_{n-1}}\eta^{\alpha_n}.$$

By rearrangement if necessary we may assume that

$$\text{(3.2)} \qquad \operatorname{Im} \tau_k(\xi) > 0, \qquad 1 \leqq k \leqq r,$$

$$\text{(3.3)} \qquad \operatorname{Im} \tau_k(\xi) < 0, \qquad r < k \leqq m_n,$$

Set

$$P_+ = \prod_{k=1}^{r} (\eta - \tau_k(\xi)), \qquad P_- = P^0/P_+.$$

Similarly we consider the boundary operators of the form

$$\text{(3.4)} \quad Q_j(D) = \sum_{\langle \beta, q\rangle \leqq p_j} b_\beta D^\beta, \qquad j = 1, \ldots, r, \quad 0 \leqq p_j \leqq (m_n - 1)\frac{m}{m_n} = m - q_n.$$

The corresponding polynomial of $Q_j(D)$ is

$$Q_j(\xi, \eta) = \sum_{\langle \beta, q\rangle \leqq p_j} b_\beta \xi_1^{\beta_1} \cdots \xi_{n-1}^{\beta_{n-1}} \eta^{\beta_n}$$

and

$$Q_j^0(\xi, \eta) = \sum_{\langle \beta, q\rangle = p_j} b_\beta \xi_1^{\beta_1} \cdots \xi_{n-1}^{\beta_{n-1}} \eta^{\beta_n}.$$

DEFINITION 3.1 (COMPLEMENTING CONDITION). We shall say that the $Q_j(D)$ $(j=1, \ldots, r)$ cover $P(D)$ when $Q_j^0(\xi, \eta)\,(j=1, \ldots, r)$ are linearly independent modulo $P_+(\xi, \eta)$ as the polynomials in η for every nonzero $\xi \in R^{n-1}$.

Let $C_0(R_+^n)$ denote the set of complex-valued functions which are infinitely differentiable in R_+^n and vanish for (x, y) with $|x|^2 + y^2$ sufficiently large. We denote by $\hat{v}(\xi, y)$ the Fourier transform of $v(x, y) \in C_0^\infty((R_+^n)^-)$ with respect to the variables $x_1, \ldots, x_{n-1}$:

$$\hat{v}(\xi, y) = (2\pi)^{-(n-1)/2} \int_{R^{n-1}} e^{-i\langle \xi, x\rangle} v(x, y)\, dx.$$

As usual we set

$$\|v\| = \left(\iint_{(R_+^n)^a} |v(x, y)|^2\, dx\, dy\right)^{1/2} = \left(\int_0^\infty \int_{|\xi| < \infty} |\hat{v}(\xi, y)|^2\, d\xi\, dy\right)^{1/2}.$$

Corresponding to the operator (3.1) we employ the notation

$$\text{(3.5)} \qquad \langle \xi \rangle = (|\xi_1|^{m_1} + \cdots + |\xi_{n-1}|^{m_{n-1}})^{1/m}.$$

For a real number p, we shall make use of the scalar product

$$\langle v_1, v_2\rangle_p = \int_{R^{n-1}} (1 + \langle \xi \rangle^2)^p \hat{v}_1(\xi, 0)\overline{\hat{v}_2(\xi, 0)}\, d\xi,\ v_1, v_2 \in C_0^\infty((R_+^n)^a).$$

The corresponding norm is given by

$$\text{(3.6)} \qquad |v|_p = (\langle v, v\rangle_p)^{1/2}.$$

Then we have the following two theorems.

THEOREM 3.1. *Let $P(D)$ and $Q_j(D)$, $j=1,\ldots,r$, be given as above and $Q_j(D)$ $(j=1,\ldots,r)$ cover $P(D)$. Then there is a constant*[1] *C depending only on P^0 and Q_j^0 $(j=1,\ldots,r)$ such that*

$$\sum_{\langle\alpha,q\rangle=m} \|D^\alpha v\| \leqq C\left(\|P^0(D)v\|+\sum_{j=1}^{r} |Q_j^0(D)v|_{m-p_j-(m/2m_n)}\right) \tag{3.7}$$

for all $v\in C_0^\infty((R_n^+)^-)$.

THEOREM 3.2. *Let $P(D)$ and $Q_j(D)$, $j=1,\ldots,r$, be the same as in Theorem 3.1. Then there is a constant C depending only on P and Q_j $(j=1,\ldots,r)$ such that*

$$\sum_{\langle\alpha,q\rangle\leqq m} \|D^\alpha v\| \leqq C\left(\|P(D)v\|+\sum_{j=1}^{r} |Q_j(D)v|_{m-p_j-(m/2m_n)}+\|v\|\right) \tag{3.8}$$

for all $v\in C_0^\infty((R_+^n)^a)$.

Theorem 3.1 can be proved by the modification of Schechter's method [12], we shall give the complete proof in the following section. The proof of Theorem 3.2 will be given in §5.

4. **Proof of Theorem 3.1.** Let $R(\xi,\eta)$ be any monomial such that

$$R(\xi,\eta) = \xi_1^{\alpha_1}\cdots\xi_{n-1}^{\alpha_{n-1}}\eta^{\alpha_n}, \qquad \langle\alpha,q\rangle = m. \tag{4.1}$$

Then there is a constant K_1 such that

$$|R(\xi,\eta)| \leqq K_1|P^0(\xi,\eta)|, \qquad (\xi,\eta)\in R^n. \tag{4.2}$$

In fact, $R(\xi,\eta)/P^0(\xi,\eta)$ is continuous on the surface $|\xi|^2+\eta^2=1$ in R^n, and to replace ξ_j by $\xi_j t^{1/m_j}$, $j=1,\ldots,n-1$ and η by $\eta t^{1/m_n}$ in (4.2) is nothing but to multiply both sides of (4.2) by t if $t>0$. Hence (4.2) is valid for every $(\xi,\eta)\in R^n$.

Now it will be convenient to make the following definition.

DEFINITION 4.1. We shall say that a function $p(\xi,\eta)$ is homogeneous of degree k (with respect to weight q), if for any $t>0$ it holds

$$p(t^{m/m_1}\xi_1, t^{m/m_2}\xi,\ldots,t^{m/m_n}\eta) = t^k p(\xi,\eta), \qquad (\xi,\eta)\in R^n. \tag{4.3}$$

We recall that $Q_j^0(\xi,\eta)=\sum_{\langle\alpha,q\rangle=p_j} b_\alpha\xi_1^{\alpha_1}\cdots\xi_{n-1}^{\alpha_{n-1}}\eta^{\alpha_n}$, $0\leqq p_j\leqq m-(m/m_n)$ that is, Q_j^0 is homogeneous of degree p_j with respect to weight q. We observe that by multiplying each $Q_j^0(\xi,\eta)$ by an appropriate power of $\langle\xi\rangle=(|\xi_1|^{m_1}+\cdots+|\xi_{n-1}|^{m_{n-1}})^{1/m}$, we may assume that each $Q_j^0(\xi,\eta)$ is homogeneous of degree $m-(m/m_n)$ with respect to weight q. Then $Q_j^0(\xi,\eta)$, $1\leqq j\leqq r$, may no longer be polynomials in the ξ, but this does not affect the following argument.

For simplicity we assume that the roots $\tau_k(\xi)$ of $P^0(\xi,z)=0$ are simple, because

(1) In this paper we use the same symbol C to express different constants.

it is easy to prove Theorem 3.1 for the general case. Resolving into partial fractions we have

$$\frac{R}{P^0} = \sum_{k=1}^{m_n} \frac{e_k}{\eta - \tau_k}, \qquad \frac{Q_j^0}{P^0} = \sum_{k=1}^{m_n} \frac{q_{jk}}{\eta - \tau_k},$$

where

$$e_k(\xi) = \frac{R(\xi, \tau_k)}{\dfrac{\partial P^0(\xi, \tau_k)}{\partial \eta}}, \qquad q_{jk}(\xi) = \frac{Q_j^0(\xi, \tau_k)}{\dfrac{\partial P^0(\xi, \tau_k)}{\partial \eta}}, \qquad 1 \leqq j \leqq r, 1 \leqq k \leqq m_n.$$

We can easily verify

$$t^{m/m_n}\tau_k(\xi) = \tau_k(t^{m/m_1}\xi_1, \ldots, t^{m/m_{n-1}}\xi_{n-1}), \qquad t > 0, 1 \leqq k \leqq m_n, \tag{4.4}$$

$$t^{m/m_n}e_k(\xi) = e_k(t^{m/m_1}\xi_1, \ldots, t^{m/m_{n-1}}\xi_{n-1}), \qquad t > 0, 1 \leqq k \leqq m_n. \tag{4.5}$$

Similarly we have

$$q_{jk}(\xi) = q_{jk}(t^{m/m_1}\xi_1, \ldots, t^{m/m_{n-1}}\xi_{n-1}), \qquad t > 0, 1 \leqq j \leqq r, 1 \leqq k \leqq m_n. \tag{4.6}$$

In particular, it follows that there are constants K_2 and K_3 such that

$$|e_k(\xi)| \leqq K_2\langle\xi\rangle^{m/m_n} = K_2\left(\sum_{j=1}^{n-1} |\xi_j|^{m_j}\right)^{1/m_n}, \qquad 1 \leqq k \leqq m_n, \tag{4.7}$$

$$K_3^{-1}\langle\xi\rangle^{m/m_n} \leqq |\mathrm{Im}\, \tau_k(\xi)| \leqq K_3\langle\xi\rangle^{m/m_n}, \qquad 1 \leqq k \leqq m_n. \tag{4.8}$$

Let $\hat{v}(\xi, y)$ be the Fourier transform of $v(x, y) \in C_0^\infty((R_+^n)^a)$ with respect to the variables $x_1, \ldots, x_{n-1}$ and define it to be zero for $y < 0$. We consider $\hat{v}(\xi, y)$ as a function of y with a vector parameter ξ. Set

$$\hat{v}(\xi, \eta) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-i\eta \cdot y}\hat{v}(\xi, y)\, dy.$$

Then we see that [recall that $D_y = i^{-1}(\partial/\partial y)$]

$$[D_y v]^\sim = \eta\hat{v} + \frac{i}{(2\pi)^{1/2}}\, \hat{v}(\xi, 0). \tag{4.9}$$

Next define

$$\begin{aligned} f(\xi, y) &= P^0(\xi, D_y)\hat{v}(\xi, y) && \text{for } y \geqq 0 \\ &= 0 && \text{for } y < 0, \end{aligned}$$

and set

$$P_k^0(\xi, \eta) = \frac{P^0(\xi, \eta)}{\eta - \tau_k(\xi)}, \qquad 1 \leqq k \leqq m_n.$$

Then by (4.9)

$$\tilde{f} = (\eta - \tau_k)[P_k^0(\xi, D_y)\hat{v}]^\sim + W_k, \qquad 1 \leqq k \leqq m_n, \tag{4.10}$$

where

$$W_k = \frac{i}{(2\pi)^{1/2}} P_k^0(\xi, D_y)\hat{v}(\xi, 0), \qquad 1 \leqq k \leqq m_n.$$

Since

$$R = \sum_{k=1}^{m_n} e_k P_k^0,$$

we have by (4.9)

$$\begin{aligned}[R(\xi, D_y)\hat{v}]^{\sim} &= \sum_{k=1}^{m_n} e_k [P_k^0(\xi, D_y)\hat{v}]^{\sim} \\ &= \sum_{k=1}^{m_n} e_k \frac{\tilde{f} - W_k}{\eta - \tau_k} \\ &= \frac{R}{P^0}\tilde{f} - \sum_{k=1}^{m_n} \frac{e_k W_k}{\eta - \tau_k}\end{aligned}$$

and hence

$$|[R(\xi, D_y)\hat{v}]^{\sim}| \leqq K_1|\tilde{f}| + K_2\langle\xi\rangle^{m/m_n} \sum_{k=1}^{m_n} \frac{|W_k|}{|\eta - \tau_k|}.$$

Thus by Parseval's formula and (4.8) we see

$$\begin{aligned}\int_0^\infty |R(\xi, D_y)\hat{v}|^2\, dy &\leqq C\left(\int_{-\infty}^{\infty} |\tilde{f}|^2\, d\eta + \langle\xi\rangle^{2m/m_n} \sum_{k=1}^{m_n} |W_k|^2 \int_{-\infty}^{\infty} \frac{d\eta}{|\eta - \tau_k|^2}\right) \\ &\leqq C'\left(\int_{-\infty}^{\infty} |\tilde{f}|^2\, d\eta + \langle\xi\rangle^{m/m_n} \sum_{k=1}^{m_n} |W_k|^2\right).\end{aligned} \tag{4.11}$$

We note that $\operatorname{Im} \tau_k < 0$ for $r < k \leqq m_n$. Paley-Wiener's theorem implies

$$\int_{-\infty}^{\infty} \frac{\tilde{f}}{\eta - \tau_k}\, d\eta = -2\pi i W_k, \qquad r < k \leqq m_n.$$

Hence by (4.8) and by Schwarz's inequality

$$|W_k|^2 \leqq \text{const.}\, \langle\xi\rangle^{-m/m_n} \int_{-\infty}^{\infty} |\tilde{f}|^2\, d\eta, \qquad r < k \leqq m_n. \tag{4.12}$$

Therefore (4.11) becomes

$$\int_0^\infty |R(\xi, D_y)\hat{v}|^2\, dy \leqq C\left(\int_{-\infty}^{\infty} |\tilde{f}|^2\, d\eta + \langle\xi\rangle^{m/m_n} \sum_{k=1}^{r} |W_k|^2\right). \tag{4.13}$$

Next we observe that there is no complex vector $\omega = (\omega_1, \ldots, \omega_r) \neq 0$ such that $\sum_{k=1}^{r} q_{jk}\omega_k = 0$, $1 \leqq k \leqq r$. For, otherwise there would be a complex vector $\lambda = (\lambda_1, \ldots, \lambda_r) \neq 0$ such that

$$\sum_{j=1}^{r} \lambda_j q_{jk} = 0, \qquad 1 \leqq k \leqq r, \tag{4.14}$$

and hence

$$P^{0-1}\sum_{j=1}^{r}\lambda_j Q_j^0=\sum_{j=1}^{r}\sum_{k=1}^{m_n}\frac{\lambda_j q_{jk}}{\eta-\tau_k}=\sum_{k=r+1}^{m_n}\sum_{j=1}^{r}\frac{\lambda_j q_{jk}}{\eta-\tau_k}$$

which shows that $\sum_{j=1}^{r}\lambda_j Q_j^0$ is a multiple of P_+. This contradicts the complementing condition (Definition 3.1). Thus the expression

$$\sum_{j=1}^{r}\left|\sum_{k=1}^{r} q_{jk}\omega_k\right|^2$$

is positive on the compact set $\sum_{k=1}^{r}|\omega_k|^2=1$, $|\xi|=1$. Hence by the homogeneity properties of the q_{jk}, there exists some constant K_4

$$\sum_{k=1}^{r}|\omega_k|^2\leqq K_4\sum_{j=1}^{r}\left|\sum_{k=1}^{r} q_{jk}\omega_k\right|^2 \tag{4.15}$$

for all ω and ξ.

Now recall that

$$Q_j^0=\sum_{k=1}^{r} q_{jk}P_k^0+\sum_{k=r+1}^{m_n} q_{jk}P_k^0,$$

$$W_k=\frac{i}{(2\pi)^{1/2}}P_k^0(\xi,D_y)\hat{v}(\xi,0).$$

By the triangle inequality

$$\left|\sum_{k=1}^{r} q_{jk}P_k^0\hat{v}(\xi,0)\right|\leqq|Q_j^0\hat{v}(\xi,0)|+\left|\sum_{k=r+1}^{m_n} q_{jk}W_k\right|(2\pi)^{1/2}.$$

Hence combining (4.12), (4.13) and (4.15) we get

$$\int_0^\infty|R(\xi,D_y)\hat{v}(\xi,y)|^2\,dy\leqq C\left(\int_{-\infty}^{\infty}|\tilde{f}|^2\,d\eta+(1+\langle\xi\rangle)^{m/m_n}\sum_{j=1}^{r}|Q_j^0\hat{v}(\xi,0)|^2\right). \tag{4.16}$$

If we now integrate with respect to ξ we obtain (3.7). This completes the proof.

5. **Proof of Theorem 3.2.** First we need the following lemma.

LEMMA 5.1. *For any $\varepsilon>0$ there exists a constant $C=C(\varepsilon)>0$ such that*

$$\sum_{\langle\beta,q\rangle<m}\|D^\beta v\|\leqq\varepsilon\sum_{\langle\alpha,q\rangle=m}\|D^\alpha v\|+C\|v\|,\qquad v\in C_0^\infty((R_+^n)^a). \tag{5.1}$$

Proof. We extend $v=v(x,y)\in C_0^\infty((R_+^n)^a)$ to the whole space R^n setting

$$\begin{aligned} v_1(x,y)&=v(x,y), &&y\geqq 0\\ &=\sum_{k=1}^{m_n}\lambda_k v(x,-ky), &&y<0,\end{aligned}$$

where the λ_k are constants chosen so that all the derivatives $D_y^j v$ for $0\leqq j\leqq m_n-1$

are continuous at $y=0$. Here λ_k depends only on m_n. We observe that for α satisfying $\langle\alpha, q\rangle \leqq m$

$$[D^\alpha v_1]^\sim = \xi_1^{\alpha_1}\cdots\xi_{n-1}^{\alpha_{n-1}}\eta^{\alpha_n}\tilde{v}_1(\xi, \eta),$$

and that, for any $\varepsilon>0$ and β ($\langle\beta, q\rangle < m$),

$$|\xi_1^{\beta_1}\cdots\xi_{n-1}^{\beta_{n-1}}\eta^{\beta_n}| \leqq \varepsilon \sum_{\langle\alpha,q\rangle=m} |\xi_1^{\alpha_1}\cdots\xi_{n-1}^{\alpha_{n-1}}\eta^{\alpha_n}|+C$$

with a constant C depending only on ε. By using Parseval's identity the lemma follows.

By Lemma 5.1 we have

$$\begin{aligned}\sum_{\langle\alpha,q\rangle\leqq m}\|D^\alpha v\| &= \sum_{\langle\alpha,q\rangle<m}\|D^\alpha v\| + \sum_{\langle\alpha,q\rangle=m}\|D^\alpha v\| \\ &\leqq (1+\varepsilon)\sum_{\langle\alpha,q\rangle=m}\|D^\alpha v\| + C(\varepsilon)\|v\|, \qquad v\in C_0^\infty((R_+^n)^a).\end{aligned}$$

So, from Theorem 3.1 we get

$$\sum_{\langle\alpha,q\rangle\leqq m}\|D^\alpha v\| \leqq C\left(\|P^0(D)v\| + \sum_{j=1}^r |Q_j^0(D)v|_{m-p_j-(m/2m_n)} + \|v\|\right), \tag{5.2}$$

$$v\in C_0^\infty((R_+^n)^a).$$

Similarly we can see that for any $\varepsilon>0$ there is a constant $C>0$ such that

$$\|P^0(D)v\| \leqq \varepsilon\sum_{\langle\alpha,q\rangle=m}\|D^\alpha v\| + C\|v\| + \|P(D)v\|, \qquad v\in C_0^\infty((R_+^n)^a).$$

Taking ε sufficiently small (for instance $\varepsilon=\frac{1}{2}$) we get the inequality

$$\sum_{\langle\alpha,q\rangle\leqq m}\|D^\alpha v\| \leqq C\left(\|P(D)v\| + \sum_{j=1}^r |Q_j^0(D)v|_{m-p_j-(m/2m_n)} + \|v\|\right). \tag{5.3}$$

It remains to replace $Q_j^0(D)v$ by $Q_j(D)v$ in (5.3). To do so, again we need to extend $v(x, y)$ to the whole space R^n as in the proof of Lemma 5.1 and we denote the extension by $v_1(x, y)$.

For any $v\in C_0^\infty((R_+^n)^-)$, we have by Schwarz's inequality

$$\begin{aligned}|\hat{v}(\xi, 0)|^2 &\leqq \frac{1}{2\pi}\left(\int_{-\infty}^\infty |\hat{v}_1(\xi, \eta)|\,d\eta\right)^2 \\ &\leqq \frac{1}{2\pi}\int_{-\infty}^\infty (1+\langle\xi\rangle^{2m/m_n}+\eta^2)|v_1(\xi, \eta)|^2\,d\eta \int_{-\infty}^\infty \frac{d\eta}{1+\langle\xi\rangle^{2m/m_n}+\eta^2}.\end{aligned}$$

The last integral is equal to $\pi(1+\langle\xi\rangle^{2m/m_n})^{-1/2}$. Hence we get

$$\begin{aligned}(1+\langle\xi\rangle)^{m-p_j-m/2m_n}|\xi^{\alpha'}D_y^{\alpha_n}\hat{v}(\xi, 0)| &\leqq C(1+\langle\xi\rangle)^{m-p_j}\left(\int|\xi^{\alpha'}\eta^{\alpha_n}\hat{v}_1|^2\,d\eta\right)^{1/2} \\ &\quad + (1+\langle\xi\rangle)^{m-p_j-m/m_n}\left(\int|\xi^{\alpha'}\eta^{\alpha_n+1}\hat{v}_1|^2\,d\eta\right)^{1/2}.\end{aligned} \tag{5.4}$$

Here we put $\alpha=(\alpha', \alpha_n)$, $\alpha'=(\alpha_1, \ldots, \alpha_{n-1})$ and we assume $\langle\alpha, q\rangle=p_j$ ($p_j\leqq m-m/m_n$). For the first term on the right hand side, the method used already gives

$$(1+\langle\xi\rangle)^{m-p_j}|\xi^{\alpha'}\eta^{\alpha_n}| \leqq C(1+|\xi_1|^{m_1}+\cdots+|\eta|^{m_n}), \qquad (\xi, \eta)\in R^n,$$

for some constant C. Similarly for the second term, we have

$$(1+\langle\xi\rangle)^{m-p_j-m/m_n}|\xi^{\alpha'}\eta^{\alpha_n+1}| \leqq C(1+|\xi_1|^{m_1}+\cdots+|\eta|^{m_n}), \qquad (\xi, \eta)\in R^n.$$

Thus by extending v to v_1 and by Parseval's formula, we get

$$|D^\alpha v|_{m-p_j-m/2m_n} \leqq C \sum_{\langle\beta,q\rangle\leqq m} \|D^\beta v\|, \qquad v\in C_0^\infty((R_+^n)^a) \tag{5.5}$$

for any α such that $\langle\alpha, q\rangle=p_j$.

By triangle inequality we have

$$|Q_j^0(D)v|_{m-p_j-m/2m_n} \leqq |Q_j(D)v|_{m-p_j-m/2m_n}+C \sum_{\langle\alpha,q\rangle<p_j} |D^\alpha v|_{m-p_j-m/2m_n}.$$

Hence a slight modification of the proof of Lemma 5.1 gives for any $\varepsilon>0$

$$\begin{aligned} |Q_j^0(D)v|_{m-p_j-m/2m_n} \leqq |Q_j(D)v|_{m-p_j-m/2m_n}&+\varepsilon \sum_{\langle\alpha,q\rangle=p_j} |D^\alpha v|_{m-p_j-m/2m_n} \\ &+C(\varepsilon)|v|_{m-p_j-m/2m_n}, \qquad v\in C_0^\infty((R_+^n)^a). \end{aligned} \tag{5.6}$$

By the argument used in the proof of the inequality (5.5) we get for the third term on the right hand side of (5.6)

$$|v|_{m-p_j-m/2m_n} \leqq \varepsilon \sum_{\langle\alpha,q\rangle=m} \|D^\alpha v\|+C(\varepsilon)\|v\|, \qquad v\in C_0^\infty((R_+^n)^a). \tag{5.7}$$

Here we may assume $p_j>0$, otherwise we do not need such an inequality because $p_j=0$ implies $Q_j=$const. and $Q_j^0=Q_j$.

Combining the inequalities (5.5), (5.6), and (5.7) we have

$$|Q_j^0(D)v|_{m-p_j-m/2m_n} \leqq |Q_j(D)v|_{m-p_j-m/2m_n}+2\varepsilon \sum_{\langle\alpha,q\rangle\leqq m} \|D^\alpha v\|+C\|v\|. \tag{5.8}$$

Finally, taking ε sufficiently small we arrive at the conclusion:

$$\sum_{\langle\alpha,q\rangle\leqq m} \|D^\alpha v\| \leqq C\left(\|P(D)v\|+\sum_{j=1}^{r} |Q_j(D)v|_{m-p_j-m/2m_n}+\|v\|\right), \qquad v\in C_0^\infty((R_+^n)^a). \tag{5.9}$$

This completes the proof of Theorem 3.2.

6. **Coerciveness inequality** (II). The case of variable coefficients. The conclusions of Theorem 3.1 and Theorem 3.2 can be extended to operators with variable coefficients.

Let Ω be a domain in R_+^n. It is supposed that the boundary of Ω contains an open set ω ($\neq\varnothing$) in the plane $y=0$. For convenience, assume the origin $(0,\ldots,0)$ is contained in the (interior of) plane boundary ω.

We consider a linear differential operator

$$P(x, y, D) = \sum_{\langle\alpha,q\rangle\leqq m} a_\alpha(x, y)D^\alpha, \tag{6.1}$$

where $a_\alpha(x, y)$ are complex valued functions defined on $\Omega \cup \omega$ and infinitely differentiable. We assume that the operator (6.1) is quasi-elliptic of weight $q=(m/m_1, \ldots, m/m_n)$ in $\Omega \cup \omega$, more precisely, there is a constant $K>0$ such that

$$|\xi_1|^{m_1}+\cdots+|\xi_{n-1}|^{m_{n-1}}+|\eta|^{m_n} \leqq K\Big|\sum_{\langle\alpha,q\rangle=m} a_\alpha(x, y)\xi^{\alpha'}\eta^{\alpha_n}\Big| \tag{6.2}$$

for any $(\xi, \eta) \in R^n$ and for any $(x, y) \in \Omega \cup \omega$. We may assume the coefficient of $D_n^{m_n}$ is identically equal to 1.

Set $P(D) = \sum_{\langle\alpha,q\rangle\leqq m} a_\alpha(0, 0)D^\alpha$ and assume that $P(D)$ is of determined type r, $1 \leqq r \leqq m_n$ (see §1). We consider r boundary operators $Q_j(x, D)$ defined on ω:

$$Q_j(x, D) = \sum_{\langle\alpha,q\rangle\leqq p_j} b_\alpha(x)D_x^{\alpha'}D_y^{\alpha_n}, \qquad j = 1, \ldots, r, 0 \leqq p_j \leqq m-m/m_n, \tag{6.3}$$

where $b_\alpha(x)$ are complex valued functions defined on ω and infinitely differentiable. Set

$$Q_j(D) = \sum_{\langle\alpha,q\rangle\leqq p_j} b_\alpha(0)D_x^{\alpha'}D_y^{\alpha_n}, \qquad j = 1, \ldots, r.$$

We denote by Ω_δ $(\delta>0)$ the hemisphere

$$\Omega_\delta = \{(x, y); y \geqq 0, |x|^2+y^2 < \delta^2\},$$

and denote by ω_δ the plane boundary of Ω_δ:

$$\omega_\delta = \{(x, 0); x < \delta\}.$$

THEOREM 6.1. *Let $P(x, y, D)$ and $Q_j(x, D), j=1, \ldots, r$ be defined as above and assume that $Q_j(D)$ $(j=1, \ldots, r)$ cover $P(D)$. Then for sufficiently small $\delta>0$ there exists a constant $C>0$ such that*

$$\sum_{\langle\alpha,q\rangle\leqq m} \|D^\alpha v\| \leqq C\Big(\|P(x, y, D)v\|+\sum_{j=1}^r |Q_j(x, D)v|_{m-p_j-m/2m_n}+\|v\|\Big), \tag{6.4}$$
$$v \in C_0^\infty(\Omega_\delta).$$

7. **Proof of Theorem 6.1.** In view of Theorem 3.2 it holds for some constant C

$$\sum_{\langle\alpha,q\rangle\leqq m} \|D^\alpha v\| \leqq C\Big(\|P(D)v\|+\sum_{j=1}^r |Q_j(D)v|_{m-p_j-m/2m_n}+\|v\|\Big), \tag{7.1}$$
$$v \in C_0^\infty((R_+^n)^a).$$

Let us write

$$P(D)v = P(x, y, D)v+[P(D)-P(x, y, D)]v.$$

Then for any $\varepsilon>0$ we can take sufficiently small $\delta>0$ such that

$$(7.2)\qquad \|P(D)v\| \leqq \|P(x,y,D)v\|+\varepsilon \sum_{\langle\alpha,q\rangle\leqq m}\|D^\alpha v\|,\qquad v\in C_0^\infty(\Omega_\delta).$$

Put $\varepsilon=1/2$. Then we have

$$(7.3)\qquad \sum_{\langle\alpha,q\rangle\leqq m}\|D^\alpha v\| \leqq 2C\left(\|P(x,y,D)v\|+\sum_{j=1}^{r}|Q_j(D)v|_{m-p_j-m/2m_n}+\|v\|\right),\qquad v\in C_0^\infty(\Omega_\delta).$$

It remains to replace $Q_j(D)$ by $Q_j(x,D)$ in (7.3). To do so, we shall prove some lemmas. We denote by $\gamma_s\in\mathfrak{S}'(R^{n-1})$ the distribution such that

$$(7.4)\qquad \hat{\gamma}_s(\xi) = (1+\xi_1^{2m_1}+\cdots+\xi_{n-1}^{2m_{n-1}})^{s/2m}$$

for real s.

Lemma 7.1 (cf. Mizohata [9]). *Let s be real and positive. Then*

$$(7.5)\qquad \|(x^\alpha\gamma_s)*\varphi\| \leqq \varepsilon(\alpha,\delta)\|\gamma_s*\varphi\|,\qquad \varphi\in C_0^\infty(R^{n-1}),\quad \alpha\neq 0.$$

Here, $\varepsilon(\alpha,\delta)$ is a constant such that $\varepsilon(\alpha,\delta)$ tends to zero when the diameter δ of the support of φ tends to zero.

Proof. It is obvious that

$$\widehat{x^\alpha\gamma_s}(\xi) = (-1)^{|\alpha|}D_\xi^\alpha\hat{\gamma}_s(\xi) \equiv P_\alpha(\xi)\hat{\gamma}_s(\xi),$$

where $P_\alpha(\xi)$ tends to zero when $|\xi|\to\infty$. Now

$$(x^\alpha\gamma_s)*\varphi(x) = (2\pi)^{-(n-1)/2}\int_{|\xi|\leqq R}\exp(i\langle x,\xi\rangle)P_\alpha(\xi)\hat{\gamma}_s(\xi)\hat{\varphi}(\xi)\,d\xi+\int_{|\xi|\geqq R}\cdots.$$

Take R sufficiently large. Then $|P_\alpha(\xi)|\leqq\varepsilon/2$, $|\xi|>R$. The L^2-norm of the second term is dominated by $\varepsilon/2\|\gamma_s*\varphi\|$, where ε will be given later. Let R be fixed as above. The L^2-norm of the first term is estimated by

$$(7.6)\qquad \max_{|\xi|\leqq R}|P_\alpha(\xi)\hat{\gamma}_s(\xi)|\left(\int_{|\xi|\leqq R}|\hat{\varphi}(\xi)|^2\,d\xi\right)^{1/2}.$$

On the other hand, it holds that

$$|\hat{\varphi}(\xi)| \leqq \int|\varphi(x)|\,dx \leqq (\text{Vol. [supp. }\varphi])^{1/2}\cdot\|\varphi\|.$$

Hence (7.6) is estimated by

$$\max|P_\alpha(\xi)\hat{\gamma}_s(\xi)|(\text{Vol. [supp. }\varphi])^{1/2}\cdot\|\varphi\|\cdot(\text{Vol. }B(R))^{1/2},$$

where $B(R)$ is a ball whose radius is R. Clearly $\|\varphi\|\leqq\|\gamma_s*\varphi\|$ for $s>0$. So the lemma is proved.

REMARK. We note that $x^\alpha \gamma_s(x) \in L^2(R^{n-1})$ if we take α such that $|\alpha|$ is sufficiently large.

LEMMA 7.2 (CF. [9]). *Let s be real, positive and $b(x) \in C^\infty(R^{n-1})$. Then*

$$\|b(x)(\gamma_s * \varphi) - \gamma_s * (b(x)\varphi)\| \leqq \varepsilon(\delta, s)\|\gamma_s * \varphi\|, \qquad \varphi \in C_0^\infty(R^{n-1}). \tag{7.7}$$

Here $\varepsilon(\delta, s)$ is a constant, which tends to zero when the diameter of the support of φ tends to zero.

Proof. The commutator, which we must estimate, is

$$b(x)\gamma_s * \varphi - \gamma_s * b(x)\varphi = \int [b(x) - b(x')]\gamma_s(x-x')\varphi(x')\, dx'. \tag{7.8}$$

We use the Taylor's formula for $b(x')$ around x:

$$b(x') - b(x) = \sum_{1 \leqq |\alpha| \leqq l-1} \frac{(x'-x)^\alpha}{\alpha!} D^\alpha b(x) + \sum_{|\alpha| = l} b_\alpha(x, x')(x'-x)^\alpha, \qquad b_\alpha(x, x') \in C^\infty.$$

Then (7.8) is written as

$$\begin{aligned} &\sum_{1 \leqq |\alpha| \leqq l-1} (-1)^\alpha \frac{D^\alpha b(x)}{\alpha!} (x^\alpha \gamma_s) * \varphi \\ &\qquad + (-1)^{l+1} \sum_{|\alpha| = l} \int b_\alpha(x, x')(x-x')^\alpha \gamma_s(x-x')\varphi(x')\, dx'. \end{aligned} \tag{7.9}$$

By Lemma 7.1 it is clear that every expression $(D^\alpha b(x)/\alpha!)(x^\alpha\gamma_s) * \varphi$ in the first term of (7.9) has the property (7.7). We take l sufficiently large so that $x^\alpha\gamma_s(x) \in L^2$, $|\alpha| = l$ (cf. Remark of Lemma 7.1). Then every term in the second summation is estimated as follows:

$$\sup |b_\alpha(x, x')| \int |(x-x')^\alpha \gamma_s(x-x')|\, |\varphi(x')|\, dx' = \sup |b_\alpha(x, x')|(|x^\alpha\gamma_s| * |\varphi|).\text{(}^2\text{)}$$

By Hausdorff-Young's inequality

$$\| |x^\alpha\gamma_s| * |\varphi| \|_{L^2} \leqq \|x^\alpha\gamma_s\|_{L^2} \cdot \|\varphi(x)\|_{L^1}.$$

Moreover

$$\|\varphi(x)\|_{L^1} \leqq (\text{Vol. [supp. } \varphi])^{1/2} \cdot \|\varphi(x)\|_{L^2}.$$

Combining these estimates we obtain the inequality (7.7)

COROLLARY 1. *Let $\varphi \in C_0^\infty(R^{n-1})$, supp. $\varphi \subset \{x; |x| \leqq \delta\} \equiv B(\delta)$ and $b(x) \in C^\infty(R^{n-1})$ with b and each $D^\alpha b$ bounded on R^{n-1}. Then*

$$\|\gamma_s * b(x)\varphi\| \leqq C|b|_k \|\gamma_s * \varphi\|, \qquad k = k(s),\ -\infty < s < \infty, \tag{7.9}$$

(2) It follows easily from our hypotheses that $b_\alpha(x, x')$ is bounded on $R^{n-1} \times R^{n-1}$.

where

$$|b|_k = \sum_{|\alpha| \leqq k} \sup_{x \in R^{n-1}} |D^\alpha b(x)|.$$

In particular, if $b(0)=0$, *then*

$$\|\gamma_s * b(x)\varphi\| \leqq \varepsilon(\delta, s)\|\gamma_s * \varphi\|, \tag{7.10}$$

where $\varepsilon(\delta, s)$ *is a constant such that* $\varepsilon(\delta, s) \to 0$ *when the diameter* δ *of the support of* φ *tends to zero.*

The proof is easy, so we omit it here.

Now let us continue the proof of Theorem 6.1. We note that the norms $|v|_s$ and $\|\gamma_s * v(x, 0)\|_{L^2(R^{n-1})}$ are equivalent to each other. Let us write

$$Q_j(D)v(x, 0) = Q_j(x, D)v + (Q_j(D) - Q_j(x, D))v, \qquad v \in C_0^\infty(\Omega_\delta).$$

If we assume that $\delta < 1$, we may (by altering the coefficients of the $b_j(x, D)$ outside the unit ball) assume that coefficients of $b_j(x, D)$ and each of its derivatives is bounded on R^{n-1}. Then the corollary stated above yields

$$\begin{aligned} |Q_j(D)v|_{m-p_j-m/2m_n} \leqq {} & |Q_j(x, D)v|_{m-p_j-m/2m_n} \\ & + \varepsilon(\delta, p_j) \sum_{\langle \alpha, q\rangle \leqq p_j} |D^\alpha v|_{m-p_j-m/2m_n}, \qquad v \in C_0^\infty(\Omega_\delta). \end{aligned} \tag{7.11}$$

As we have seen in the proof of Theorem 3.2 there exists a constant C not depending on δ such that

$$\sum_{\langle \alpha, q\rangle \leqq p_j} |D^\alpha v|_{m-p_j-m/2m_n} \leqq C \sum_{\langle \alpha, q\rangle \leqq m} \|D^\alpha v\|, \qquad v \in C_0^\infty(\Omega_\delta). \tag{7.12}$$

Hence if we take δ sufficiently small, we get the estimate:

$$\sum_{\langle \alpha, q\rangle \leqq m} \|D^\alpha v\| \leqq C\left(\|P(x, y, D)v\| + \sum_{j=1}^{r} |Q_j(x, D)v|_{m-p_j-m/2m_n} + \|v\|\right), \quad v \in C_0^\infty(\Omega_\delta),$$

which completes the proof.

REMARK. We note that the necessity of our complementing condition (Definition 3.1) for the validity of the estimate (3.8) can be proved by the analogous way in [1, §10].

8. **Regularity at the boundary.**

We introduce function spaces Φ^m and H^s.

DEFINITION 8.1. By $\Phi^m(R_+^n) = \Phi^m$ we mean the set of all $u \in L^2(R_+^n)$ such that $D^\alpha u \in L^2(R_+^n)$ for any α satisfying $\langle \alpha, q\rangle \leqq m$. The space Φ^m is a Hilbert space with norm

$$\|u\|_m^2 = \sum_{\langle \alpha, q\rangle \leqq m} \|D^\alpha u\|^2. \tag{8.1}$$

We denote by $H^s(R^{n-1}) = H^s$ (for a real $s \geqq 0$) the set of all $u \in L^2(R^{n-1})$ such that

$$(1+\langle \xi\rangle)^s \hat{u}(\xi) \in L^2(R^{n-1}).$$

The space H^s is a Hilbert space with norm

$$|u|_s = \|(1+\langle\xi\rangle)^s\hat{u}(\xi)\|_{L^2(R^{n-1})} \cong \|\gamma_s * u\|_{L^2(R^{n-1})}, \tag{8.2}$$

where we also denote by $\hat{u}(\xi)$ the Fourier transform of u with respect to variables $x=(x_1,\dots,x_{n-1})$.

DEFINITION 8.2. We shall say $u\in\Phi^m_{\mathrm{loc}}(\Omega_\delta)$ if $\varphi u\in\Phi^m$ for all $\varphi\in C_0^\infty(\Omega_\delta)$. Similarly we shall say $u\in H^s_{\mathrm{loc}}(\omega_\delta)$ if $\varphi u(x)\in H^s$ for all $\varphi\in C_0^\infty(\omega_\delta)$.

Concerning with the properties of the spaces Φ^m and H^s we have the following lemmas.

LEMMA 8.1. *Let s and t be two integers such that $0<s<t$. Then for any $\varepsilon>0$ there exists a constant $C=C(\varepsilon)$ such that*

$$\|u\|_s \leqq \varepsilon\|u\|_t + C\|u\|, \qquad u\in\Phi^t. \tag{8.3}$$

Proof. Clearly $C_0^\infty((R^n_+)^a)$ is dense in Φ^t for every t. Hence Lemma 8.1 can be reduced to Lemma 5.1.

LEMMA 8.2. *For every $u\in\Phi^m$ there exists the trace*

$$D^\alpha u(x,0)\in H^{m-p-m/2m_n}, \qquad 0\leqq p\leqq m-m/m_n, \langle\alpha,q\rangle\leqq p,$$

and there is a constant $C=C(m)$ such that

$$\sum_{\langle\alpha,q\rangle\leqq p} |D^\alpha u|_{m-p-m/2m_n} \leqq C\|u\|_m, \qquad u\in\Phi^m. \tag{8.4}$$

The proof of Lemma 8.2 can be given by the argument stated at the end of §5. So we omit the detail.

Now take $\rho=\rho(x)\in C_0^\infty(R^{n-1})$ satisfying the following:

(i) $\int_{R^{n-1}}\rho(x)\,dx=1$ and

(ii) $\rho(\xi)=O(|\xi|^k)$, $\xi\to 0$ for some sufficiently large integer k.

Set $\rho_\varepsilon(x)=\varepsilon^{-(n-1)}\rho(x/\varepsilon)$ $(\varepsilon>0)$. If $v\in\Phi^m$, the regularization of v

$$v_\varepsilon(x,y) = \varepsilon^{-(n-1)}\int v(x',y)\rho(x-x'/\varepsilon)\,dx', \qquad y\geqq 0,$$

is infinitely differentiable with respect to x and obviously $v_\varepsilon\in\Phi^m$, and $v_\varepsilon\to v$ in $\Phi^m(R^n_+)$ as $\varepsilon\to 0$.

As usual we denote by $\mathfrak{S}((R^n_+)^a)$ the set of all the functions, each of which is a restriction in $(R^n_+)^a$ of a function in $\mathfrak{S}(R^n)$.

LEMMA 8.3. *If $u\in L^2(R^n_+)$ and $a\in\mathfrak{S}((R^n_+)^a)$ and if $\rho\in C_0^\infty(R^{n-1})$ satisfies* (i) *and* (ii), *then*

$$\|\gamma_1 * (au_\varepsilon-(au)_\varepsilon)\|_{L^2(R^n_+)}\to 0 \qquad \text{as} \quad \varepsilon\to 0. \tag{8.5}$$

The lemma is derived by a simple modification of the proof of Theorem 2.4.3 in [5], so we omit the proof. (For the definition of γ_1, see (7.4).)

THEOREM 8.1. *Consider the boundary problem*

$$P(x, y, D)u = f \quad in \quad \Omega, \tag{8.6}$$

$$Q_j(x, D)u = g_j \quad on \quad \omega, \qquad j = 1, \ldots, r, \tag{8.7}$$

where $P(x, y, D)$ is a quasi-elliptic operator given in §6, *and the boundary operators $Q_j(x, D)$ are also given in* §6.

Suppose $u \in \Phi^m_{loc}(\Omega)$. If $f \in C^\infty(\bar{\Omega})$ and if $g_j \in C^\infty(\omega), j = 1, \ldots, r$, then there exists a positive number $\delta = \delta(t)$ such that $u \in \Phi^{m+t}_{loc}(\Omega_\delta)$ for $t = 1, 2, \ldots$.

Proof. First take any tangential derivative D^β_x (here $\beta = (\beta_1, \ldots, \beta_{n-1}, 0)$). We want to prove that there exists a $\delta = \delta(\beta)$ such that

$$D^\beta_x(\varphi u) \in \Phi^m \tag{8.8}$$

for any $\varphi \in C^\infty_0(\Omega_\delta)$.

To do so, it is sufficient to prove that for any positive integer s there exists a $\delta = \delta(s)$ such that

$$\gamma_s * (\varphi u) \in \Phi^m \tag{8.9}$$

for any $\varphi \in C^\infty_0(\Omega_\delta)$. We consider the case $s = 1$.

We set $P(D) = \sum_{\langle\alpha, q\rangle \leqq m} a_\alpha(0, 0)D^\alpha$ and $Q_j(D) = \sum_{\langle\alpha, q\rangle \leqq p_j} b_\alpha(0)D^\alpha$, $j = 1, \ldots, r$, the same as in §6. As $\gamma_1 * (\varphi u)_\varepsilon \in \Phi^m(R^n_+)$ for any $\varphi \in C^\infty_0(\Omega)$ and for any $\varepsilon > 0$, the coerciveness estimate yields

$$\begin{aligned} \|\gamma_1 * (\varphi u)_\varepsilon\|_m \leqq C(&\|P(D)(\gamma_1 * (\varphi u)_\varepsilon)\| \\ &+ \sum_{j=1}^{r} |Q_j(D)\gamma_1 * (\varphi u)_\varepsilon|_{m - p_j - m/2m_n} + \|\gamma_1 * (\varphi u)_\varepsilon\|), \end{aligned} \tag{8.10}$$

where the constant C is independent of $\varphi \in C^\infty_0(\Omega)$ and of ε (>0).

Write

$$\begin{aligned} \gamma_1 * P(D)(\varphi u)_\varepsilon = \; & \gamma_1 * P(x, y, D)(\varphi u)_\varepsilon \\ & + \gamma_1 * \{(P(D) - P(x, y, D))(\varphi u)_\varepsilon\}. \end{aligned}$$

Then, for the second term in the right hand side, we can easily see by using Corollary 1 in §7, that for any $\nu > 0$ there exist $\delta = \delta(\nu)$ and $\varepsilon_0 = \varepsilon_0(\nu)$ such that

$$\|\gamma_1 * (P(D) - P(x, y, D))(\varphi u)_\varepsilon\|_{L^2(R^n_+)} \leqq \nu\|\gamma_1 * (\varphi u)_\varepsilon\|_m$$

for any $\varphi \in C^\infty_0(\Omega_\delta)$ and for any ε ($0 < \varepsilon \leqq \varepsilon_0$). Similarly we can get

$$|\gamma_1 * \{Q_j(D) - Q_j(x, D)(\varphi u)_\varepsilon\}|_{m - p_j - m/2m_n} \leqq \nu\|\gamma_1 * (\varphi u)_\varepsilon\|_m,$$

$j = 1, \ldots, r$, for any $\varphi \in C^\infty_0(\Omega_\delta)$ and for any ε ($0 < \varepsilon \leqq \varepsilon_0$). Here also δ and ε_0 depend only on ν.

Thus we get the following estimate:

$$
\begin{aligned}
(8.11)\qquad \|\gamma_1 * (\varphi u)_\varepsilon\|_m \leqq C(&\|\gamma_1 * P(x, y, D)(\varphi u)_\varepsilon\| \\
&+\sum_{j=1}^{r} |\gamma_1 * Q_j(x, D)(\varphi u)_\varepsilon|_{m-p_j-m/2m_n} \\
&+\|\gamma_1 * (\varphi u)_\varepsilon\|), \qquad \varphi \in C_0^\infty(\Omega_\delta), \quad 0 < \varepsilon \leqq \varepsilon_0,
\end{aligned}
$$

where the constant C is independent of φ and ε.

Next we prove that the norms

$$\|\gamma_1 * P(x, y, D)(\varphi u)_\varepsilon\|$$

and

$$|\gamma_1 * Q_j(x, D)(\varphi u)_\varepsilon|_{m-p_j-m/2m_n}, \qquad j = 1, \ldots, r,$$

are uniformly bounded with respect to ε $(0<\varepsilon\leqq\varepsilon_0)$. By Lemma 8.3 it is sufficient to prove the uniform boundedness of the norms

$$\|\gamma_1 * (P(x, y, D)(\varphi u)_\varepsilon\|$$

and

$$|\gamma_1 * (Q_j(x, D)(\varphi u)_\varepsilon)|_{m-p_j-m/2m_n}, \qquad j = 1, \ldots, r.$$

We note that $\gamma_1 * (P(x, y, D)(\varphi u))_\varepsilon = (\gamma_1 * P(x, y, D)(\varphi u))_\varepsilon$. If we write

$$(8.12)\qquad P(x, y, D)(\varphi u) = \varphi P(x, y, D)u + \sum_{\beta\neq 0} \frac{D^\beta\varphi}{\beta!}\cdot P^{\beta}(x, y, D)u,$$

then the fact $\varphi P(x, y, D)u = \varphi f \in C_0^\infty(\Omega_\delta)$ implies $\gamma_1 * \varphi f \in L^2(R^n_+)$. Now, we take any term $(D^\beta\varphi/\beta!)a_\alpha(x, y)D^{\alpha-\beta}u$ in the summation of (8.12). Take $\psi \in C_0^\infty(\Omega_\delta)$ such that $\psi = 1$ in supp. φ. Then

$$(D^\beta\varphi/\beta!)a_\alpha(x, y)D^{\alpha-\beta}u = (D^\beta\varphi/\beta!)a_\alpha(x, y)D^{\alpha-\beta}(\psi u).$$

By virtue of Lemma 7.2 it is sufficient to prove $\gamma_1 * D^{\alpha-\beta}(\psi u) \in L^2(R^n_+)$. We can easily observe that if $\langle\alpha, q\rangle \leqq m$, $\beta\leqq\alpha$, $\beta\neq 0$, then there exists a constant C such that

$$
\begin{aligned}
(1+|\xi_1|^{m_1/m}+\cdots+|\xi_{n-1}|^{m_{n-1}/m})&|\xi^{\alpha'-\beta'}\eta^{\alpha_n-\beta_n}| \\
&\leqq C(1+|\xi_1|^{m_1}+\cdots+|\xi_{n-1}|^{m_{n-1}}+|\eta|^{m_n})
\end{aligned}
$$

for any $(\xi, \eta) \in R^n$. Let $(\psi u)_1$ be the extension of ψu in $\Phi^m(R^n)$ by the method used in the proof of Lemma 5.1. Then $\gamma_1 * D^{\alpha-\beta}(\psi u)_1 \in L^2(R^n)$. Therefore we see $\gamma_1 * D^{\alpha-\beta}(\psi u) \in L^2(R^n_+)$. Hence we have

$$\gamma_1 * P(x, y, D)(\varphi u) \in L^2(R^n_+).$$

Now it is obvious that

$$(\gamma_1 * P(x, y, D)(\varphi u))_\varepsilon = \gamma_1 * (P(x, y, D)(\varphi u))_\varepsilon$$

is uniformly bounded for ε $(0<\varepsilon\leqq\varepsilon_0)$.

For the norms $|\gamma_1 * Q_j(x, D)(\varphi u)|_{m-p_j-m/2m_n}$, we again write

$$Q_j(x, D)(\varphi u) = \varphi Q_j(x, D)u + \sum_{\beta\neq 0} \frac{D^\beta \varphi}{\beta!} Q_j^\beta(x, D)u.$$

Similarly to the above, it is sufficient to prove that

$$|\gamma_1 * D^{\alpha-\beta}(\psi u)|_{m-p_j-m/2m_n} < \infty$$

for any $\psi \in C_0^\infty(\Omega_\delta)$, where $\langle \alpha, q\rangle \leqq p_j$, $\beta \leqq \alpha$, $\beta \neq 0$. Clearly, it holds that

$$|\gamma_1 * D^{\alpha-\beta}(\psi u)|_{m-p_j-m/2m_n} \leqq |D^{\alpha-\beta}(\psi u)|_{m-(p_j-1)-m/2m_n},$$

and $\langle \alpha-\beta, q\rangle \leqq p_j - 1$. So Lemma 8.2 gives us

$$|\gamma_1 * D^{\alpha-\beta}(\psi u)|_{m-p_j-m/2m_n} < \infty.$$

Thus, by using Lemma 8.3, we can conclude that the norms

$$\|\gamma_1 * (P(x, y, D)(\varphi u)_\varepsilon)\|$$

and

$$|\gamma_1 * Q_j(x, D)(\varphi u)_\varepsilon|_{m-p_j-m/2m_n}, \qquad j = 1, \ldots, r,$$

are uniformly bounded with respect to ε $(0<\varepsilon\leqq\varepsilon_0)$. By the estimate (8.11), we can see that the norm $\|\gamma_1 * (\varphi u)\|_m$ is uniformly bounded with respect to ε $(0<\varepsilon\leqq\varepsilon_0)$. Hence the theorem of Banach-Saks implies $\gamma_1 * \varphi u \in \Phi^m$.

We can repeat this procedure and get

$$D_x^\beta u \in \Phi^m_{\mathrm{loc}}(\Omega_\delta), \qquad |\beta| \leqq N, \quad \beta_n = 0, \quad \delta = \delta(N)$$

for any integer N. Now the equation $P(x, y, D)u = f$ can be written in the form

$$D_y^{m_n} u = -\sum_{\langle\alpha,q\rangle\leqq m;\, \alpha_n<m_n} a_\alpha(x, y) D_x^{\alpha'} D_y^{\alpha_n} u + f. \tag{8.13}$$

Differentiating (8.13) with respect to x-variables we see

$$D_x^\beta D_y^{m_n} u \in L^2_{\mathrm{loc}}(\Omega_\delta).$$

Moreover we have

$$D_x^\beta D_y^{m_n+1} u \in L^2_{\mathrm{loc}}(\Omega_\delta).$$

We can repeat this procedure and arrive at the conclusion of Theorem 8.1.

9. **Hypo-analyticity at the boundary.** In the following we shall investigate the more precise estimates of the derivatives of the solutions of a quasi-elliptic boundary problem. However, we are limited to the case of simple boundary operators.

Consider a quasi-elliptic operator $P(D)$ of weight $q = (m/m_1, \ldots, m/m_{n-1}, m/m_n)$ and of determined type r $(1\leqq r\leqq m_n)$ given by (3.1):

$$P(D) = D_y^{m_n} + \sum_{\langle\alpha,q\rangle\leqq m;\, \alpha_n<m_n} a_\alpha D_x^{\alpha'} D_y^{\alpha_n},$$

where a_α are complex constants.

As the boundary operators we take only normal derivatives such that $Q_j(D)=D_y^{k_j}$, $j=1,\ldots,r$, where $k_i\neq k_j$ if $i\neq j$ and $0\leqq k_j\leqq m_n-1$. Clearly these r boundary operators cover $P(D)$ (see Definition 3.1).

DEFINITION 9.1. Let Ω be a domain in R^n and set $d=(d_1,\ldots,d_n)$, $d_i\geqq 1$, $i=1,\ldots,n$. We call u a function of the class $G(d,\Omega)$ if u is infinitely differentiable in Ω and if for each compact set K in Ω there exist two constants C_0, C_1 such that

$$\|D^\alpha u, K\|_\infty \leqq C_0 C_1^{|\alpha|}\prod_{i=1}^n \alpha_i^{d_i\alpha_i} \tag{9.1}$$

or equivalently (if some $\alpha_i=0$, then we define $\alpha_i^{\alpha_i}=1$ and $|\alpha|^{|\alpha|}=1$ if $|\alpha|=0$)

$$\|D^\alpha u, K\|_\infty \leqq C_0 C_1^{|\alpha|}\alpha_i^{\alpha_i d_i} \tag{9.2}$$

for any α, where $\|w, K\|_\infty$ means the maximum of $|w|$ in K.

As in §6, let Ω be a domain in R^n and let the boundary of Ω contain an open set ω ($\neq\varnothing$) in the plane $y=0$.

Now we can state our results.

THEOREM 9.1 (CF. [3]). *Let $P(D)$ be the quasi-elliptic operator defined as above and take the boundary operators $D_y^{k_j}$ ($j=1,\ldots,r$) given above. Consider the boundary problem*

$$P(D)u(x,y) = f(x,y) \quad in \quad \Omega, \tag{9.3}$$

$$D_y^{k_j}u(x,0) = 0, \qquad j=1,\ldots,r \quad on \quad \omega \tag{9.4}$$

with $f\in G(\lambda q;\Omega\cup\omega)$, $\lambda q=(\lambda q_1,\ldots,\lambda q_n)$, $\lambda\geqq 1$. Then any function $u\in\Phi_{\mathrm{loc}}^m(\Omega\cup\omega)$ satisfying (9.3) *and* (9.4) *is a function in* $G(\lambda q;\Omega\cup\omega)$.

THEOREM 9.2. *Let $P(D)$ be the same as in Theorem* 9.1. *Assume that a function $u\in C^\infty(\Omega\cup\omega)$ satisfies the following conditions*:

(i) *For each compact set K in $\Omega\cup\omega$ there exist two constants C_0, C_1 such that*

$$\|P^k(D)u, K\|_\infty \leqq C_0C_1^k(mk)^{\lambda k} \qquad (\lambda\geqq 1). \tag{9.5}$$

(ii) *On the plane boundary ω, it holds that*

$$D_y^{k_j}P^k(D)u(x,0) = 0, \qquad k=0,1,2,\ldots, \tag{9.6}$$

where k_j, $j=1,\ldots,r$ are the same as in Theorem 9.1. *Then $u\in G(\lambda q;\Omega\cup\omega)$.*

REMARK. As a special case of Theorem 8.1 we see that any solution $u\in\Phi_{\mathrm{loc}}^m(\Omega\cup\omega)$ of the problem (9.3), (9.4) is infinitely differentiable on $\Omega\cup\omega$.

10. **Proof of Theorem 9.1.**

To prove Theorems 9.1 and 9.2 we make use of the methods in [3], [2] and [8].

First we shall derive some preliminary lemmas by making use of the coerciveness inequalities proved in the foregoing paragraphs.

We denote by W the set of all functions $v \in C_0^\infty(\Omega \cup \omega)$ which satisfy boundary conditions (9.4).

LEMMA 10.1. *Let $P(D)$ be that in Theorem 9.1. Then for any $\varepsilon > 0$ and for any $v \in W$ the following inequality holds*:

$$\sum_{\langle\alpha,q\rangle \leqq m} \|D^\alpha v\| \varepsilon^{\langle\alpha,q\rangle} \leqq C\{\varepsilon^m \|P(D)v\| + (1+\varepsilon^m)\|v\|\}, \tag{10.1}$$

where the constant C is independent of ε and $v \in W$.

Proof. First we see that, if $\langle\beta, q\rangle \leqq m$, then

$$|\xi^{\beta'}\eta^{\beta_n}| \leqq C\left(\sum_1^{n-1} |\xi_j|^{m_j} + |\eta|^{m_n}\right)^{\langle\beta,q\rangle/m}, \qquad (\xi, \eta) \in R^n, \tag{10.2}$$

where the constant C is independent of such β. In fact, replacing ξ_j by $\xi_j t^{m/m_j}$, $j = 1, \ldots, n-1$ and η by $\eta t^{m/m_n}$ in (10.2) means multiplying both sides by t if $t > 0$. Hence (10.2) is valid. Next noting that $\langle\beta, q\rangle/m \leqq 1$ we can easily verify that, if $\langle\beta, q\rangle \leqq m$ then

$$|\xi^{\beta'}\eta^{\beta_n}| \varepsilon^{\langle\beta,q\rangle} \leqq C\left\{\varepsilon^m\left(\sum_{j=1}^{n-1} |\xi_j|^{m_j} + |\eta|^{m_n}\right) + 1\right\} \tag{10.3}$$

for another constant C independent of such β and of $(\xi, \eta) \in R^n$. By the same way in the proof of Lemma 5.1, we get for any α, $\langle\alpha, q\rangle \leqq m$ and for any $v \in W$

$$\|D^\alpha v\| \varepsilon^{\langle\alpha,q\rangle} \leqq C\left\{\varepsilon^m \sum_{j=1}^{n} \|D_j^{m_j} v\| + \|v\|\right\}. \tag{10.4}$$

Finally we apply the coerciveness estimate (3.8) and we get Lemma 10.1.

LEMMA 10.2 (CF. [3], [10]). *For every compact set $K \subset (R_+^n)^a$ and for every h, $0 < h \leqq 1$, there are a function $\psi = \psi_{K,h}$ and constants C_α independent of h such that $\psi \in C_0^\infty(K_h)$, $\psi = 1$ on K and*

$$\|D^\alpha \psi\|_\infty \leqq C_\alpha h^{-\langle\alpha,q\rangle} \quad \textit{for every } \alpha, \tag{10.5}$$

where $K_h = \{x \in (R_+^n)^a; \operatorname{dis}(x, K) \leqq h\}$. It may be assumed that $D_y^i \psi(x, 0) = 0$, $i = 1, \ldots, m_n$.

Now we introduce some notation. For convenience we assume that the plane boundary ω contains the origin $(0, \ldots, 0)$. We denote by V the hemisphere $\{(x, y); |x|^2 + y^2 < R^2, y > 0\}$ included in Ω and put $V_{-r} = \{(x, y); |x|^2 + y^2 < (R-r)^2, y > 0\}$, $0 < r < R \leqq 1$. We set for arbitrary $l \geqq 0$,

$$\|D^\beta u; l + \langle\beta, q\rangle, V\| = \sup_{0<r<R} r^{l+\langle\beta,q\rangle} \|D^\beta u, V_{-r}\|, \tag{10.6}$$

and

$$\|u; q, \mu; l, V\| = \sup_{\beta \geqq 0;\, \beta_n = 0} \prod_{i=1}^{n-1} \left(\frac{\mu_i}{\beta_i + 1}\right)^{\beta_i q_i} \|D^\beta u; l + \langle\beta, q\rangle, V\|, \tag{10.7}$$

$$\mu = (\mu_1, \ldots, \mu_{n-1}),\ \mu_i > 0,\ 1 \leqq i \leqq n-1.$$

The following lemma is essential in our proofs of Theorem 9.1 and Theorem 9.2.

LEMMA 10.3 (CF. [2]). *Let $P(D)$ be the same as in Theorem* 9.1, *that is*, $P(D)$ *is quasi-elliptic of weight q and of determined type r. Then*

$$\sum_{\langle\alpha,q\rangle\leqq m} \|D^\alpha u, V_{-r}\|\varepsilon^{\langle\alpha,q\rangle} \leqq C\Big\{\varepsilon^m\|P(D)u, V_{-(r-\delta)}\| + \varepsilon^m\delta^{-m} \sum_{\langle\alpha,q\rangle<m} \delta^{\langle\alpha,q\rangle}\|D^\alpha u, V_{-(r-\delta)}\| + \|u, V_{-(r-\delta)}\|\Big\} \tag{10.8}$$

for any $\varepsilon>0$ and for any $u\in C^\infty(V)$ satisfying the boundary condition (9.4). *The constant C is independent of r, δ $(0<\delta<r, 0<r<+\infty)$ and u.*

Proof. Take $\psi=\psi_{V_{-r,\delta}}$ defined in Lemma 10.2. Then ψu also satisfies the boundary condition (9.4). Hence ψu satisfies the inequality (10.1). So

$$\sum_{\langle\alpha,q\rangle\leqq m} \|D^\alpha u, V_{-r}\|\varepsilon^{\langle\alpha,q\rangle} \leqq C\{\varepsilon^m\|P(D)(\psi u)\| + (1+\varepsilon^m)\|\psi u\|\}.$$

By the Leibniz formula we have

$$P(D)(\psi u) = \psi P(D)u + \sum_{\langle\alpha,q\rangle\leqq m}\ \sum_{\beta_i\leqq\alpha_i;\,\beta\neq 0} a_\alpha C_{\alpha,\beta}D^{\alpha-\beta}uD^\beta\psi,$$

where $C_{\alpha,\beta}$ are appropriate constants. Hence we have

$$\begin{aligned}\|P(D)(\psi u)\| &\leqq C\|P(D)u, V_{-(r-\delta)}\| + C\sum_{\langle\alpha,q\rangle\leqq m}\ \sum_{\beta_i\leqq\alpha_i;\,\beta\neq 0}\delta^{-\langle\beta,q\rangle}\|D^{\alpha-\beta}u, V_{-(r-\delta)}\|,\\ &\leqq C\|P(D)u, V_{-(r-\delta)}\| + C\sum_{\langle\gamma,q\rangle<m}\delta^{-m+\langle\gamma,q\rangle}\|D^\gamma u, V_{-(r-\delta)}\|.\end{aligned}$$

Thus the lemma is proved.

Now in (10.8) we put $\varepsilon=\chi\cdot t\cdot r$, $\delta=rt$ with sufficiently small χ, t (>0) determined later. Then (10.8) turns into

$$\begin{aligned}\sum_{\langle\alpha,q\rangle\leqq m} \|D^\alpha u, V_{-r}\|\chi^{\langle\alpha,q\rangle}(tr)^{\langle\alpha,q\rangle} \leqq C\Big\{&\chi^m(tr)^m\|P(D)u, V_{-r(1-t)}\|\\ &+\chi^m\sum_{\langle\alpha,q\rangle<m}(rt)^{\langle\alpha,q\rangle}\|D^\alpha u, V_{-r(1-t)}\|\\ &+\|u, V_{-r(1-t)}\|\Big\}.\end{aligned} \tag{10.9}$$

Multiplying both sides by $(tr)^l$ $(l\geqq 0)$ we have

$$\begin{aligned}\sum_{\langle\alpha,q\rangle\leqq m} &\|D^\alpha u, V_{-r}\|r^{l+\langle\alpha,q\rangle}\chi^{\langle\alpha,q\rangle}t^{\langle\alpha,q\rangle+l}\\ &\leqq C\Big\{\|P(D)u, V_{-r(1-t)}\|(r(1-t))^{l+m}\chi^m\left(\frac{t}{1-t}\right)^{l+m}\\ &\quad+\chi^m\sum_{\langle\alpha,q\rangle<m}\|D^\alpha u, V_{-r(1-t)}\|(r(1-t))^{l+\langle\alpha,q\rangle}\left(\frac{t}{1-t}\right)^{l+\langle\alpha,q\rangle}\\ &\quad+\|u, V_{-r(1-t)}\|(r(1-t))^l\left(\frac{t}{1-t}\right)^l\Big\}.\end{aligned}$$

Hence we get by (10.6)

$$\sum_{\langle\alpha,q\rangle\leqq m} \|D^\alpha u; l+\langle\alpha, q\rangle, V\|\chi^{\langle\alpha,q\rangle}t^{l+\langle\alpha,q\rangle}$$

$$\leqq C\left\{\|P(D)u; l+m, V\| \left(\frac{t}{1-t}\right)^{l+m}\cdot\chi^m+\chi^m\right.$$

$$\left.\sum_{\langle\alpha,q\rangle<m} \|D^\alpha u; l+\langle\alpha, q\rangle, V\| \left(\frac{t}{1-t}\right)^{l+\langle\alpha,q\rangle}+\|u; l, V\|\left(\frac{t}{1-t}\right)^{l}\right\}.$$

Now assume $0<t\leqq 1/(l+m)$. Then there is a constant $c>0$ such that

$$(t/(1-t))^{l+\langle\alpha,q\rangle} \leqq t^{l+\langle\alpha,q\rangle}e^c$$

for any α satisfying $\langle\alpha, q\rangle\leqq m$. Taking χ sufficiently small here we get, for another constant C,

$$\sum_{\langle\alpha,q\rangle\leqq m} \|D^\alpha u; l+\langle\alpha, q\rangle, V\|t^{\langle\alpha,q\rangle} \leqq C\{\|Pu; l+m, V\|t^m+\|u; l, V\|\}$$

with any t such that $0<t\leqq 1/(l+m)$.

We note that in Lemma 10.3 the terms $\varepsilon^{\langle\alpha,q\rangle}$ and $\delta^{\langle\alpha,q\rangle}$ can be replaced by $\varepsilon^{\langle\alpha,\lambda q\rangle}$ and $\delta^{\langle\alpha,\lambda q\rangle}$ with any $\lambda\geqq 1$ respectively. Thus we can obtain the following lemma.

LEMMA 10.4. *There exists a constant C such that*

$$\sum_{\langle\alpha,q\rangle\leqq m} \|D^\alpha u; l+\langle\alpha, \lambda q\rangle, V\|t^{\langle\alpha,\lambda q\rangle} \leqq C\{\|P(D)u; l+m, V\|t^{m\lambda}+\|u; l, V\|\} \tag{10.10}$$

for all $u\in C(\Omega\cup\omega)$ satisfying the condition (9.4), *provided that* $0<t\leqq 1/(l+m)$, $\lambda\geqq 1$.

By making use of these lemmas we can prove Theorems 9.1 and 9.2. For simplicity we consider the case $\lambda=1$. It will be convenient to use the notation:

$$(D^\beta P(D)u)_t = t^{\langle\beta,q\rangle+m}D^\beta P(D)u, \qquad (D^\beta u)_t = t^{\langle\beta,q\rangle}D^\beta u; \tag{10.11}$$

$$\begin{aligned} B_0(D^\beta u) &= \|(D^\beta u)_t; l+\langle\beta, q\rangle, V\|^{(3)}, \\ B_{i+1}(u) &= \max_{\langle\beta,q\rangle\leqq m;\,\beta_n=0} B_i(D^\beta u), \qquad i\geqq 0; \end{aligned} \tag{10.12}$$

$$\begin{aligned} B_0(D^\beta Pu) &= \|(D^\beta Pu)_t; l+\langle\beta, q\rangle+m, V\|, \\ B_{i+1}(Pu) &= \max_{\langle\beta,q\rangle\leqq m;\,\beta_n=0} B_i(D^\beta Pu), \qquad i\geqq 0; \end{aligned} \tag{10.13}$$

$$\begin{aligned} \|u; q, \mu; l, V\| &= \sup_{\beta\leqq 0;\,\beta_n=0} \prod_{i=1}^{n-1} \left(\frac{\mu_i}{\beta_i}\right)^{\beta_i q_i} \|D^\beta u; l+\langle\beta, q\rangle, V\|. \\ \|Pu; q, \mu; l, V\| &= \sup \prod_{i=1}^{n-1} \left(\frac{\mu_i}{\beta_i}\right)^{\beta_i q_i} \|D^\beta pu; l+m+\langle\beta, q\rangle, V\|. \end{aligned} \tag{10.7}$$

(3) The B_i are functions of t also.

LEMMA 10.5. *There is a constant* $C>1$ *such that*

$$C^{-j}B_j(u) \leqq \max\left\{\max_{1\leqq k\leqq j} C^{k-j}B_{j-k}(Pu),\ B_0(u)\right\} \tag{10.14}$$

for $j=1,2,\dots$ *and for all* $u\in C^\infty(V)$ *satisfying* (9.4), *provided that* $0<t \leqq 1/(l+jm)$.

Proof. We see that (10.10) means

$$B_1(u) \leqq \max\{CB_0(Pu),\ CB_0(u)\} \tag{10.15}$$

with some positive constant C. The inequality (10.15) shows that (10.14) is true when $j=1$ and $0<t\leqq 1/(l+m)$. If we replace u by $D^\beta u$ in (10.8), multiply both sides by $\varepsilon^{\langle\beta,q\rangle}$, and then proceed as in the proof of Lemma 10.4, we find that

$$B_2(u) \leqq \max\{CB_1(Pu),\ CB_1(u)\},$$

provided that $0<t\leqq 1/(l+2m)$. Again by (10.15) we obtain

$$B_2(u) \leqq \max\{CB_1(Pu),\ C^2B_0(Pu),\ C^2B_0(u)\}, \tag{10.16}$$

provided that $0<t\leqq 1/(l+2m)$. Proceeding in this way, we can prove (10.14) for all j.

LEMMA 10.6. *Let* B_0 *be defined by* (10.12), (10.13) *with* $t_j=1/(l+jm)$ *for* l *and* j *fixed. Then there are constants* $c<1$ *and* C_1, *independent of* j, *such that*

$$C_1^{-1}\|u;q,c\mu;l,V\| \leqq C^{-j}B_j(Pu,t_j)+B_0(u) \tag{10.17}$$

and

$$C^{-j}B_j(Pu,t_j) \leqq \|Pu;q,\mu;l,V\|, \tag{10.18}$$

where $\mu=(C^{1/q_1},\dots,C^{1/q_{n-1}})$.

Proof. Put $N=\|Pu;q,\mu;l,V\|$, where $\mu=(C^{1/q_1},\dots,C^{1/q_{n-1}})$, and suppose that $t=1/(l+jm)$. Then

$$\begin{aligned}\max C^{-j}B_j(Pu) &\leqq \max_{\langle\beta,q\rangle\leqq jm;\,\beta_n=0} C^{-|\beta|}\left(\frac{1}{l+jm}\right)^{m+\langle\beta,q\rangle}\|D^\beta Pu;l+m+\langle\beta,q\rangle,V\|\\ &\leqq \max_{\langle\beta,q\rangle\leqq jm;\,\beta_n=0} C^{-|\beta|}\left(\frac{1}{l+jm}\right)^{m+\langle\beta,q\rangle}\prod_{i=1}^{n-1}\left(\frac{\beta_i}{\mu_i}\right)^{\beta_iq_i}N\\ &\leqq \max_{\langle\beta,q\rangle\leqq jm;\,\beta_n=0}\prod_{i=1}^{n-1}\left(\frac{\beta_i}{l+jm}\right)^{\beta_iq_i}\left(\frac{1}{l+jm}\right)^{m}N \leqq N.\end{aligned}$$

This proves (10.18).

Next, with the same t, μ as above we have, for c determined later,

$$C^{-j}B_j(u) \geqq \max_{(j-1)m\leqq\langle\beta,q\rangle\leqq jm} C^{-j}\left(\frac{1}{l+jm}\right)^{\langle\beta,q\rangle} \|D^\beta u; l+\langle\beta,q\rangle, V\|$$

$$\geqq \max_{(j-1)m\leqq\langle\beta,q\rangle\leqq jm} C^{-j}\left(\frac{1}{l+jm}\right)^{\langle\beta,q\rangle} \prod_{i=1}^{n-1}\left(\frac{\beta_i}{c\mu_i}\right)^{\beta_i q_i}\prod_{i=1}^{n-1}\left(\frac{c\mu_i}{\beta_i}\right)^{\beta_i q_i}$$

$$\cdot \|D^\beta u; l+\langle\beta,q\rangle, V\|$$

$$\geqq C' \max_{(j-1)m\leqq\langle\beta,q\rangle\leqq jm} \prod_{i=1}^{n-1}\left[\frac{\beta_i}{c(l+jm)}\right]^{\beta_i q_i}\prod_{i=1}^{n-1}\left(\frac{\mu_i}{\beta_i}\right)^{\beta_i q_i} \|D^\beta u; l+\langle\beta,q\rangle, V\|,$$

where C' and C_0 are constants independent of j. Put

$$K = \prod_{i=1}^{n-1}\left[\frac{\beta_i}{c(l+jm)}\right]^{\beta_i q_i}, \qquad (j-1)m \leqq \langle\beta,q\rangle \leqq jm.$$

Then

$$(C'K)^{-1} = (C')^{-1}\prod_{i=1}^{n-1}\left[\frac{c(l+jm)}{\beta_i+1}\right]^{\beta_i q_i} \leqq C_1 c^{-\langle\beta,q\rangle}, \qquad (j-1)m \leqq \langle\beta,q\rangle \leqq jm,$$

is finite if c is sufficiently small. This proves (10.17).

Finally we obtain the following estimate:

$$\|u; q, c\mu; l, V\| \leqq C\{\|Pu; q, \mu; l, V\| + \|u; l, V\|\}, \tag{10.19}$$

for all $u \in C^\infty(V)$ satisfying (9.4). In the same way we can get the estimate of the type

$$\|D^\alpha u; q, c\mu; l+\langle\alpha,q\rangle, V\| \leqq C\{\|Pu; q, \mu; l, V\| + \|u; l, V\|\} \tag{10.20}$$

for any α such that $\langle\alpha,q\rangle \leqq m$.

Now we can complete the proof of Theorem 9.1. Let $f(x,y)$ be in $G(q, \Omega\cup\omega)$. Then for any hemisphere $K=\{(x,y); |x|^2+y^2\leqq r, y\geqq 0\}\subset V$, there are constants C_0, C_1 such that

$$\|D^\alpha f, K\| \leqq C_0 C_1^{|\alpha|}|\alpha|^{\langle\alpha,q\rangle} \tag{10.21}$$

for all α.

By the inequality (10.20) we have for new constants C_0, C_1

$$\|D_x^\beta D^\alpha u, K\| \leqq C_0 C_1^{|\beta|}|\beta|^{\langle\beta,q\rangle} \tag{10.22}$$

for any β ($\beta_n=0$) and α ($\langle\alpha,q\rangle\leqq m$). We may assume that the corresponding constants in (10.21) and (10.22) are the same.

The equation $P(D)u=f$ can be written in the form

$$D_y^{m_n}u = f - \sum_{\langle\alpha,q\rangle\leqq m;\, \alpha_n<m_n} a_\alpha D^\alpha u. \tag{10.23}$$

We note that $\langle\alpha,q\rangle\leqq m$ and $\alpha_n=m_n$ imply $\alpha_i=0$, $(i=1,\ldots,n-1)$, and that for any positive integer $k\,(\leqq m_n)$, $\langle\alpha, q\rangle \leqq m$ and $\alpha_n = m_n - k$ imply $\sum_{i=1}^{n-1} \alpha_i q_i \leqq k q_n$. Differentiating (10.23) with respect to x-variables and applying (10.21) and (10.22), we have

$$\|D_x^\beta D_y^{m_n} u, K\| \leqq C_0 C_1^{|\beta|} |\beta|^{\langle\beta, q\rangle} + B C_0 C_1^{|\beta|} |\beta|^{\langle\beta, q\rangle}, \tag{10.24}$$

where we put $B = 1 + \sum_\alpha |a_\alpha|$. Again differentiating (10.23) we have

$$\begin{aligned} D_x^\beta D_y^{m_n+1} u = D_x^\beta D_y f - &\sum_{\langle\alpha, q\rangle \leqq m;\, \alpha_n = m_n - 1} a_\alpha D_x^\beta D_x^{\alpha'} D_y^{m_n} u \\ &+ \sum_{\langle\alpha, q\rangle \leqq m;\, \alpha_n \leqq m_n - 2} a_\alpha D_x^\beta (D_x^{\alpha'} D_y^{\alpha_n + 1} u) \end{aligned}$$

where we put $a_\alpha = 0$ when $\alpha_n < 0$. Applying (10.24) and Lemma 10.1 we have

$$\begin{aligned} \|D_x^\beta D_y^{m_n+1} u, K\| \leqq\ & C_0 C_1^{|\beta|+1} (|\beta|+1)^{\langle\beta, q\rangle + q_n} \\ & + B(B+1) C_0 C_1^{|\beta| + q_n} (|\beta| + q_n)^{\langle\beta, q\rangle + q_n} \\ & + B C_0 C_1^{|\beta| + q_n + m} (|\beta| + q_n + m)^{\langle\beta, q\rangle + q_n + m}. \end{aligned} \tag{10.25}$$

Repeating the procedure we can obtain, by a simple induction argument on k,

$$\begin{aligned} \|D_x^\beta D_y^{m+k} u, K\| \leqq (B+1)^{k+1} C_0 C_1^{|\beta| + k q_n + m} (|\beta| + k q_n + m)^{\langle\beta, q\rangle + k q_n + m}&, \\ k = 1, 2, \ldots&. \end{aligned} \tag{10.26}$$

Hence we can take new constants C_0, C_1 such that

$$\|D_x^\beta D_y^{m_n+k} u, K\| \leqq C_0 C_1^{|\beta| + m_n + k} (|\beta| + m_n + k)^{\langle\beta, q\rangle + (m_n + k) q_n} \tag{10.27}$$

for any β $(\beta_n = 0)$ and for any $k \geqq 0$.

Applying Sobolev's lemma to the inequality (10.27) we obtain Theorem 9.1.

11. **Proof of Theorem 9.2.** From the estimate (10.10) we easily obtain

$$\begin{aligned} \max_{\langle\alpha, q\rangle \leqq m;\, \langle\beta, q\rangle \leqq km;\, \beta_n = 0} & \|D^{\beta+\alpha} u; l + \langle\beta + \alpha, \lambda q\rangle, V\| t^{\langle\beta+\alpha, \lambda q\rangle} \\ & \leqq C^k \max_{0 \leqq j \leqq k} \|P^j u; l + jm, V\| t^{\lambda j m} \end{aligned} \tag{11.1}$$

for all $u \in C^\infty(\Omega \cup \omega)$ satisfying (9.6), provided that $0 < t \leqq 1/(l + km)$, $k = 0, 1, 2, \ldots$.

In a quite similar manner to that used in the proof of Lemma 10.6 we obtain the estimate of the form

$$\max_{\langle\alpha, q\rangle \leqq m} \|D^\alpha u; \lambda q, \mu; l + \langle\alpha, \lambda q\rangle, V\| \leqq C \sup_k \left(\frac{\mu}{mk}\right)^{m\lambda k} \|P^k u; l + \lambda k m, V\|, \tag{11.2}$$

where

$$\|D^\alpha u; \lambda q, \mu; l + \langle\alpha, \lambda q\rangle, V\| = \sup_{\beta \geqq 0;\, \beta_n = 0} \prod_{i=1}^{n-1} \left(\frac{\mu_i}{\beta_i}\right)^{\beta_i q_i} \|D^{\beta+\alpha} u; l + \beta + \langle\alpha, \lambda q\rangle, V\|.$$

The same argument as in the end of the proof of Theorem 9.1 completes the proof of Theorem 9.2.

REMARK. The conclusion of Theorems 9.1 and 9.2 can be extended to operators with variable coefficients. The proof can be obtained by a quite similar argument to the proof of Theorem 9.1 and Theorem 9.2 (cf. [2], [8]).

Added in Proof. The conclusion of Theorems 9.1 and 9.2 can be extended to the general quasi-elliptic boundary problems defined in §3. Details will be given in a future publication.

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